Analysis 2

16 April 2024

Warm-up: Next slide.

Find f'(x), also written df/dx.

Differentiate the functions whose letters are the start of your first or last name.

A) $x^2 - 5x + 27$

B) $\frac{1}{2} - x$

C) cx^3

D) $8\sin(x)$

E) $7\cos(x)$

F) $x^2\cos(x)$

G) $6x^{-2}$

H) 1238

I) $\sqrt[3]{x}$

J) $x\cos(x)$

K) $5 - x^3$

L) $(x^2 + 1)(x^{10} - 3)$

Ł) $\frac{2}{\sqrt{x}}$

M) $\cos(x) + \sqrt{x}$

N) $x^{-1/9}$

O) $\sqrt{\sqrt{x}}$

Ö) $7x^2 + 5 + 3x^{-1}$

P) $x^4 - x^3 + x^2 - x + 1$

Q) $5 + \sqrt{5}$

R) $3\sin(x) + 2\cos(x)$

S) $\frac{-2}{x^5}$

T) $\cos(x) \cdot \sqrt{x}$

U) $\cos(x) \cdot \sin(x)$

V) $\sqrt{x^5}$

Y) $6x^{-2} + 5x^2$

Z) x^{100}

Find f' or df/dx. Do this now.

Differentiate the functions whose letters are the start of your first or last name.

A) $2x - 5$

B) -1

C) $3c\,x^2$

D) $8\cos(x)$

E) $-7\sin(x)$

F) $2x\cos(x) - x^2\sin(x)$

G) $-12x^{-3}$

H) 0

I) $\frac{1}{3}x^{-2/3}$

J) $\cos(x) - x\sin(x)$

K) $-3x^2$

L) $(x^2+1)10x^9 + 2x(x^{10}-3)$
$= 12x^{11} + 10x^9 - 6x$

Ł) $\frac{-1}{x^{3/2}}$

M) $-\sin(x) + \frac{1}{2}x^{-1/2}$

N) $\frac{-1}{9}x^{-10/9}$

O) $\frac{1}{4}x^{-3/4}$

Ö) $14x - 3x^{-2}$

P) $4x^3 - 3x^2 + 2x - 1$

Q) 0

R) $3\cos(x) - 2\sin(x)$

S) $\frac{10}{x^6}$

T) $\frac{\cos(x)}{2\sqrt{x}} - \sin(x)\sqrt{x}$

U) $\cos(x)^2 - \sin(x)^2$

V) $\frac{5}{2}x^{3/2}$

Y) $-12x^{-3} + 10x$

Z) $100x^{99}$

Higher derivatives

The **second derivative** of a function is the derivative of its derivative.

We can write this as

- f'' because it is $(f')'$, or
- $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ because it is $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\mathrm{d}}{\mathrm{d}x}[f]\right]$.

Speaking:
"f double-prime"

To calculate a second derivative, just differentiate *twice*!

Example: for $f(x) = 9x^4$ we have $f''(x) = 108x^2$ because

$$9x^4 \quad \rightsquigarrow \quad 36x^3 \quad \rightsquigarrow \quad 108x^2.$$

Higher derivatives

The **third derivative** of a function is the derivative of its second derivative.

We can write this as

- f''' because it is $(f'')'$, or
- $\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$ because it is $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right]$.

Speaking:
"f triple-prime"

To calculate a second derivative, just differentiate *three times*!

Example: for $f(x) = 9x^4$ we have $f'''(x) = 216x$ because

$$9x^4 \rightsquigarrow 36x^3 \rightsquigarrow 108x^2 \rightsquigarrow 216x.$$

Higher derivatives

We know $f'(x)$ can tell us whether $f(x)$ is increasing or decreasing.

What can $f''(x)$ tell us?

- Next week

What can $f'''(x)$ tell us?

- It's not used often.

If $f(t)$ is position then
$f'(t)$ is velocity or speed,
$f''(t)$ is acceleration,
$f'''(t)$ is jerk,
$f^{(4)}(t)$ is snap or jounce,
$f^{(5)}(t)$ is crackle,
$f^{(6)}(t)$ is pop.

Higher derivatives

Task: Find the second derivative of $x^2 \cos(x)$.

Note: Writing

$$x^2 \cos(x) = -x^2 \sin(x) + 2x \cos(x)$$

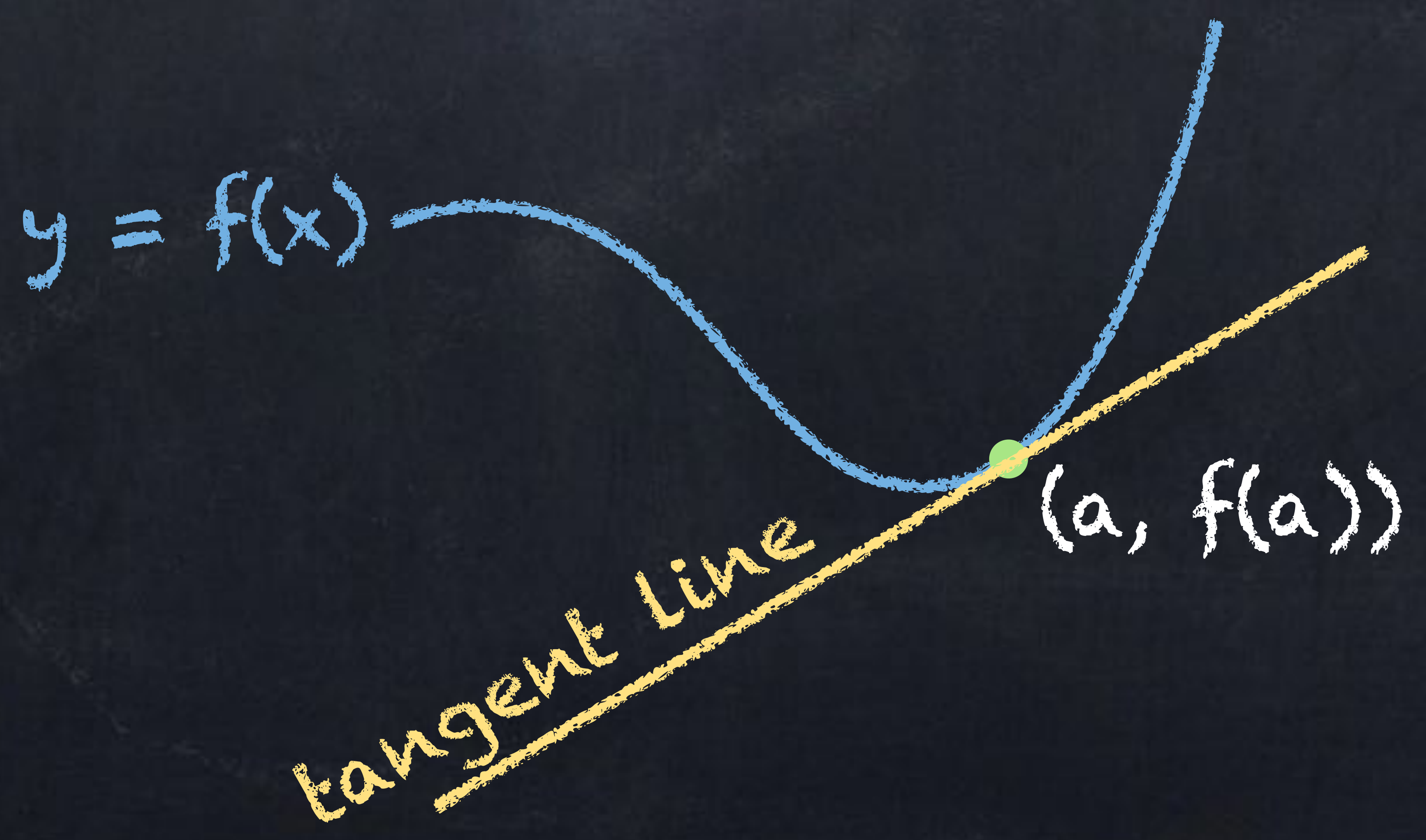
is incorrect, and you will lose points for it.

Also, do not write $f(x)'$. It should be $f'(x)$.

The **tangent line to the curve $y = f(x)$ at $x = a$** is the line through the point $(a, f(a))$ whose slope is the number $f'(a)$.

- Using "slope-intercept form", $y = mx + b$, we have $m = f'(a)$ but will have to do a bit of extra calculation to find b.
- Using "slope-point" format, $y - y_0 = m \cdot (x - x_0)$, is much easier. The tangent line will always* have the equation

$$y = f(a) + f'(a)(x - a).$$



*unless the tangent line is vertical

Last time

Critical points

A number c in the domain of $f(x)$ is a critical point of $f(x)$ if either $f'(c) = 0$ (horizontal tangent line) or $f'(c)$ doesn't exist (vertical tangent line, or jump, or corner).

Increasing and decreasing

On an interval or at a single point:

- If $f' > 0$ then f is increasing (↗️).
- If $f' < 0$ then f is decreasing (↘️).

Minimum and maximum

- How do these relate to derivatives?

Finding local extremes

This year we will not do tasks with *absolute* extremes from formulas.

But we will need to find *local* extremes for $f(x)$ from its formula.

What can we say about f' at different points in these pictures?

Finding local extremes

To find the local min/max of $f(x)$,

1. Find the critical points of f.

2. Compute signs of f' somewhere in between each CP, and at one point with $x <$ all CP, and at one point with $x >$ all CP.

3. **The First Derivative Test**

- If $f' > 0$ to the left of $x = c$ and $f' < 0$ to the right of $x = c$, then f has a local maximum at $x = c$.
- If $f' < 0$ to the left of $x = c$ and $f' > 0$ to the right of $x = c$, then f has a local minimum at $x = c$.
- If f' has the same sign on both sides of $x = c$, then f has *neither* a local minimum nor local maximum at $x = c$.

Example 1: Given that the critical points of

$$g(x) = x^4 + \frac{4}{3}x^3 - 10x^2 + 12x$$

are -3 and 1, classify each as a local minimum, local maximum, or neither.

Summary of rules:

- $(c)' = 0$
- $(x^c)' = c\,x^{c-1}$
- $(a^x)' = ???$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\ln x)' = ???$
- $(cf)' = c\,(f')$
- $(f+g)' = f' + g'$
- $(fg)' = fg' + f'g$

We do not need to use f and g as the names of the functions, and we do not need to use x as the variable.

- If $u = 10x^3 + 1$ then $\frac{du}{dx} = 30x^2$.
- If $u = t\cos(t)$ then $\frac{du}{dt} = \cos(t) - t\sin(t)$.
- If $y = \sin(v)$ then $\frac{du}{dv} = \cos(v)$.
- If $f = u^2$ then $\frac{df}{du} = 2u$.

We have seen how to do derivatives of $f(x) + g(x)$ and $f(x) - g(x)$ and $f(x) \cdot g(x)$. We will look at $\frac{f(x)}{g(x)}$ later.

There is one other important way to combine functions: the **composition** of $f(x)$ and $g(x)$ is the function $f(g(x))$, which can also be written as $f \circ g$.

Examples of compositions:

- $\sin(x^2)$
- $\sqrt{x^2 + 1}$
- $\ln(x^3 + 8)$
- $e^{(-x^2)}$
- $(5 + \cos(x))^3$
- $\sqrt{\sin(x^2)}$

Before learning the general formula for $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right]$, let's look at a composition that we can already differentiate with other methods:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(10x^3+1)^2\right] = ?$$

We will answer this three different ways:

- By expanding $(10x^3+1)^2 = 100x^6 + 20x^3 + 1$.
- By the PRODUCT RULE because $(10x^3+1)^2 = (10x^3+1)\cdot(10x^3+1)$.
- By the CHAIN RULE (new)!

Although $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not really a fraction, the idea of canceling out parts of fractions is a nice way to remember one of the official Chain Rule formulas.

Chain Rule

- $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$
- $f(g(x))' = f'(g(x)) \cdot g'(x)$

You do *not* need to know *any* of these formulas.

You only need to be able to *use* the Chain Rule to find derivatives of functions.

Differentiate $(\sin(x))^4$.

Chain Rule

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

For the example $(\sin(x))^4$, we call $\sin(x)$ the "inside function" and we call $(\ )^4$ the "outside function".

For any differentiable function g, we have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(g(x)\right)^4\right] = 4\left(g(x)\right)^3 \cdot g'(x)$.

Using the Product Rule or the Chain Rule, we can see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(g(x)\right)^2\right] = 2\,g(x)\,g'(x).$$

Task 1: Find the derivative of $(4x^2 - 8x + 9)^{50}$.

Task 2: Find the derivative of $\sin(4x^2 - 8x + 9)$.

Task 3: Find the derivative of $(3x - 7)\cos(x)$.

Task 4: Find the derivative of $x^3 e^x + \sin(x^2)$.

- Use the SUM RULE first.
- Then use the PRODUCT RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3 e^x\right]$.
- And use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x^2)\right]$.

Task 5: Differentiate $\cos(7x^3 + e^{12x}\sin(\pi x))$.

- CHAIN RULE first.
- Then SUM. Then……

Task 6a: Find the derivative of $(3x - 7)(2x + 1)^5$.

- Use the PRODUCT RULE first.
- Then use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x + 1)^5\right]$.

Task 6b: Differentiate $(3x - 7)(2x + 1)^{-1}$.

- Use the PRODUCT RULE first.
- Then use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x + 1)^{-1}\right]$.

Name

Simplify the formula above as much as possible.

Task 6b: Differentiate $(3x-7)(2x+1)^{-1}$.

- Use the PRODUCT RULE first.
- Then use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^{-1}\right]$.

This is one way to differentiate $\dfrac{3x-7}{2x+1}$. There is also “the quotient rule”.

The Quotient Rule $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$ can be helpful, but you can always use Product and Chain instead, like we did for $\frac{3x-7}{2x+1} = (3x-7)(2x+1)^{-1}$.

Example: Find the derivative of $\tan(x)$.

- You should know that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Derivative formulas

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	(later)
$\ln(x)$	(later)

You should memorize these!

$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\tan(x)$	$\sec(x)^2$

Maybe these too.

Derivative formulas

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	(later)
$\ln(x)$	(later)

Constant Multiple: $(cf)' = cf'$

Sum Rule: $(f+g)' = f' + g'$

Product Rule:

$$(fg)' = fg' + f'g$$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Chain Rule:

$$\big(f(g)\big)' = f'(g) \cdot g'$$